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Dynamics in the Sherrington–Kirkpatrick Model. I. The First Step

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We study properties of the random configuration $\{s_j(1)\}_{j=1}^N$ produced by the first step of the parallel dynamics in the Sherrington–Kirkpatrick model. We show that the law of large numbers holds for the sequence of overlaps between the initial (nonrandom) configuration $\{s_j(0)\}_{j=1}^N$ and $\{s_j(1)\}_{j=1}^N$, and obtain the distribution of the fluctuations around the limiting value. As a by-product we derive the average number of the fixed points $\{s_j(1)\}_{j=1}^N = \{s_j(0)\}_{j=1}^N$ with a given value of the magnetization $m_N = (1/N) \sum_{j=1}^N s_j(0)$.

KEY WORDS: Spin-glass dynamics; fixed points; fluctuations.

1. INTRODUCTION

The zero-temperature parallel dynamics of the Sherrington–Kirkpatrick (SK) model[5] is commonly defined by the rule

$$s_j(t+1) = \operatorname{sgn}\left[\frac{1}{\sqrt{N}} \sum_{k(\neq j)=1}^{N} J_{jk} s_k(t) + \frac{J_0}{N} \sum_{k(\neq j)=1}^{N} s_k(t)\right], \qquad j = 1,..., N \quad (1)$$

where $J_{jk} = J_{kj}$ for $j > k$; $s_j \in \{-1, 1\}$, $j = 1, 2,..., N$, are the Ising spins, and $\{J_{jk}\}_{j<k}$ are independent random variables with the standard normal distribution.

Among the extensive literature devoted to the study of the dynamics of the SK model the paper by Gardner *et al.*[4] deserves special mention. In that paper the values of some macroscopic observables were calculated for the first four steps of the dynamics (1) in the case $J_0 = 0$. Another series of papers where exact results for the SK and related models were obtained (see, e.g., refs. 3

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and 8) was initiated by the paper of Tanaka and Edwards,[7] who calculated the average number of fixed point of the dynamics (1) in the case $J_0 = 0$.

In the present paper we undertake an investigation of (only) the first step of the dynamics (1). That is, we investigate the properties of the sequence of random variables (more precisely, of the triangular array)

$$s_j(1) = \operatorname{sgn}\left[\frac{1}{\sqrt{N}} \sum_{k(\neq j)=1}^{N} J_{jk} s_k(0) + \frac{J_0}{N} \sum_{k(\neq j)=1}^{N} s_k(0)\right], \qquad j = 1,..., N \tag{2}$$

in the limit $N \to \infty$. In particular, in Section 2 we calculate the probability $\Pr[\mathbf{s}_N | \boldsymbol{\sigma}_N]$ to obtain a configuration $\mathbf{s}_N \equiv \{s_j\}_{j=1}^N$ at $t = 1$ given a configuration $\boldsymbol{\sigma}_N \equiv \{\sigma_j\}_{j=1}^N$ at $t = 0$ [see Eq. (10)]. Using these transition probabilities, we show in Section 3 that the law of large numbers holds for the sequence of arithmetic means

$$q_N = \frac{1}{N} \sum_{j=1}^{N} s_j(1)\, s_j(0) \tag{3}$$

and find the distribution of the fluctuations around the limiting value [see Eqs. (18) and (19)]. In the particular case $\boldsymbol{\sigma}_N = \mathbf{s}_N$ the transition probability $\Pr[\mathbf{s}_N | \boldsymbol{\sigma}_N]$ is the probability that the configuration $\boldsymbol{\sigma}_N$ is a fixed point of the dynamics (1). Therefore, having found the transition probabilities, we calculate in Section 4 the average number of the fixed points as a by-product [see Eq. (20)].

The method we use in the present paper is somewhat different from (but certainly is closely related to) those used in refs. 4, 7, and 8. Our method, however, in some cases allows us to avoid many potentially dangerous steps (like unjustified inversion of the integration order, introduction of delta functions of a complex argument, formal application of the saddle point method for evaluation of multiple integrals, etc.) frequently used by physicists (see, e.g., ref. 8).

Following the suggestion of a referee, the size of this paper was significantly reduced at the expense of details of calculations. The complete version of the paper is available from the author by request.

2. TRANSITION PROBABILITIES

We choose as the initial condition for the dynamics (1) a configuration $\{\sigma_j\}_{j=1}^{\infty}$ about which we assume only that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \sigma_j = m_*$$

To calculate the transition probabilities $\Pr[\mathbf{s}_N|\boldsymbol{\sigma}_N]$ [that is, the probability to have a configuration $\{s_j\}_{j=1}^N$ at time $t=1$ given that $\{s_j(0)\}_{j=1}^N = \{\sigma_j\}_{j=1}^N$] we calculate first the joint probability density $f(\boldsymbol{\mu}) \equiv f(\mu_1;...;\mu_N)$ of the "local energies" $s_j h_j(0)$, where

$$h_j(0) \equiv \frac{1}{\sqrt{N}} \sum_{k(\neq j)=1}^{N} J_{jk}\sigma_k + \frac{J_0}{N} \sum_{k(\neq j)=1}^{N} \sigma_k, \qquad j=1,..., N$$

One has

$$f(\boldsymbol{\mu}) = \mathbf{E} \prod_{j=1}^{N} \delta(s_j h_j(0) - \mu_j)$$

where $\mathbf{E}(\cdot)$ is the expectation with respect to the distribution of $\{J_{jk}\}_{j<k}^N$ and $\delta(\cdot)$ is the Dirac delta function. Using the integral representation for the delta function

$$\delta(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau\, e^{i\tau\mu}$$

and exchanging the order of $\mathbf{E}(\cdot)$ and integration over τ's, we obtain

$$\begin{aligned} f(\boldsymbol{\mu}) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\tau_1 \cdots d\tau_N \\ \times \exp\left[i \sum_{j=1}^{N} \tau_j \left(\mu_j - \frac{J_0 s_j}{N} \sum_{k(\neq j)=1}^{N} \sigma_k \right) \right] \\ \times \mathbf{E} \exp\left[-\frac{i}{\sqrt{N}} \sum_{j=1}^{N} \tau_j s_j \sum_{k(\neq j)=1}^{N} J_{jk}\sigma_k \right] \end{aligned} \tag{4}$$

Calculating the expectation in Eq. (4) (the interaction strength symmetry $J_{jk} = J_{kj}$ must be taken into account), we obtain

$$\begin{aligned} f(\boldsymbol{\mu}) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\tau_1 \cdots d\tau_N \\ \times \exp\left[i \sum_{j=1}^{N} \tau_j \left(\mu_j - \frac{J_0 s_j}{N} \sum_{k(\neq j)=1}^{N} \sigma_k \right) \right] \\ \times \exp\left\{ -\frac{1}{2N} \left[(N-2) \sum_{j=1}^{N} \tau_j^2 + \left(\sum_{j=1}^{N} \tau_j \sigma_j s_j \right)^2 \right] \right\} \end{aligned} \tag{5}$$

So far our calculations follow closely the commonly used derivation (see, e.g., refs. 8 and 7), the next step of which would be the linearization of the quadratic term in

$$\exp\left\{-\frac{1}{2N}\left(\sum_{j=1}^{N} \tau_j \sigma_j s_j\right)^2\right\}$$

using a well-known integral identity. We, however, chose another way, which is (in our opinion) a more efficient one. Note that the integrand in Eq. (5) is an exponential of a symmetric quadratic form, and hence the integral can be easily calculated after diagonalization of the quadratic form by an orthogonal transformation. The elements of the matrix $\hat{M}$ associated with the quadratic form are given by

$$M_{j,k} = (N-2)\,\delta_{j,k} + \sigma_j s_j \sigma_k s_k$$

Hence the matrix $\hat{M}$ has two eigenvalues $\lambda_1 = 2(N-1)$ (nondegenerate) and $\lambda_2 = N-2$ ($N-1$ times degenerate). The corresponding eigenvectors are $\mathbf{v}_1 = N^{-1/2}\{\sigma_j s_j\}_{j=1}^{N}$ and all vectors orthogonal to $\mathbf{v}_1$.

Note now that the term in the argument of the exponential in Eq. (5) linear in τ's can be written as a linear combination of $\mathbf{v}_1$ and a vector $\mathbf{v}_2$ orthogonal to $\mathbf{v}_1$,

$$\sum_{j=1}^{N} \tau_j\left(\mu_j - \frac{J_0 s_j}{N} \sum_{k(\neq j)=1}^{N} \sigma_k\right) \equiv \boldsymbol{\tau}\cdot\mathbf{u} = \alpha_1 \boldsymbol{\tau}\cdot\mathbf{v}_1 + \alpha_2 \boldsymbol{\tau}\cdot\mathbf{v}_2$$

where

$$\alpha_1 = \frac{1}{\sqrt{N}} \sum_{l=1}^{N} \sigma_l s_l \left(\mu_l - \frac{J_0 s_l}{N} \sum_{k(\neq l)=1}^{N} \sigma_k\right)$$

is the scalar product of $\mathbf{u}$ and $\mathbf{v}_1$,

$$\alpha_2 = \left[\sum_{l=1}^{N}\left(\mu_l - \frac{J_0 s_l}{N} \sum_{k(\neq l)=1}^{N} \sigma_k\right)^2 - \alpha_1^2\right]^{1/2}$$

is the norm of $\mathbf{u} - \alpha_1 \mathbf{v}_1$, and $\mathbf{v}_2$ is the vector obtained by normalization of $\mathbf{u} - \alpha_1 \mathbf{v}_1$. Since the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, the vector $\mathbf{v}_2$ is an eigenvector of the matrix $\hat{M}$. Consider now a basis $\{\mathbf{v}_k\}_{k=1}^{N}$, where $\{\mathbf{v}_k\}_{k=3}^{N}$ is an arbitrary orthonormal set orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$. Introducing the new integration variables $y_k = \boldsymbol{\tau} \cdot \mathbf{v}_k$, $k = 1,..., N$, we diagonalize the quadratic form in Eq. (5) and obtain

$$f(\boldsymbol{\mu}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^{N} \frac{dy_k}{2\pi} \times \exp\left[i(\alpha_1 y_1 + \alpha_2 y_2) - \left(1 - \frac{1}{N}\right) y_1^2 - \frac{1}{2}\left(1 - \frac{2}{N}\right) \sum_{k=2}^{N} y_k^2 \right] \tag{6}$$

Now the integration over $y_1,..., y_N$ can be easily carried out,

$$f(\boldsymbol{\mu}) = \sqrt{\frac{N-2}{2(N-1)}} \left[2\pi \left(1 - \frac{2}{N}\right)\right]^{-N/2} \times \exp\left[\frac{\alpha_1^2}{4(1-1/N)(1-2/N)} - \frac{1}{2(1-2/N)} \times \sum_{l=1}^{N} \left(\mu_l - \frac{J_0 s_l}{N} \sum_{k(\neq l)=1}^{N} \sigma_k \right)^2 \right]$$

Thus, the joint distribution of the "local energies" $s_j h_j(0)$ is virtually the same as the Gibbs distribution in the Gaussian Curie–Weiss model in the high-temperature (paramagnetic) phase.

Having found the probability density of the "local energies," one can calculate by integration the probability to obtain the configuration $\{s_j\}_{j=1}^{N}$ at $t = 1$, which is equal to the probability of all "local energies" being positive:

$$\Pr[\mathbf{s}_N | \boldsymbol{\sigma}_N] = \int_0^{\infty} \cdots \int_0^{\infty} \prod_{k=1}^{N} d\mu_k \, f(\boldsymbol{\mu})$$

To perform the integrations over μ's we proceed with the standard method for solving the Curie–Weiss model, namely, we use the identity

$$\exp\left(\frac{ax^2}{2}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \exp\left[-\frac{1}{2} t^2 + \sqrt{a}\, tx \right]$$

to decouple the integration variables. Introducing the notations

$$N_{1,2} \equiv N\nu_{1,2} \equiv \#\{k\colon s_k = 1;\ \sigma_k = \pm 1\} \tag{7}$$

we obtain

$$\Pr[\mathbf{s}_N | \boldsymbol{\sigma}_N] = \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{\infty} dt \exp\left[-N\left(\frac{1}{2} t^2 - \Phi_N(t; \nu_1; \nu_2) \right) \right] \tag{8}$$

where

$$\Phi_N(t; \nu_1; \nu_2) = \nu_1 \log I_1 + \left(\frac{1+m_N}{2} - \nu_1\right) \log(1 - I_1)$$
$$+ \nu_2 \log I_2 + \left(\frac{1-m_N}{2} - \nu_2\right) \log(1 - I_2)$$
$$I_{1,2} = \left(1 - \frac{2}{N}\right)^{-1/2} \int_0^\infty \frac{d\mu}{\sqrt{2\pi}} \exp\left[-\frac{(\mu - J_0 m_N \mp t \pm J_0/N)^2}{2(1-2/N)}\right]$$
$$m_N = \frac{1}{N} \sum_{j=1}^{N} \sigma_j \tag{9}$$

Despite the fact that the function $\Phi_N(t; \nu_1; \nu_2)$ depends on N, the evaluation of the integral in Eq. (8) requires only minor modifications of the Laplace method (see, e.g., ref. 1, where the Laplace method was applied in a similar situation), which yields

$$\Pr[\mathbf{s}_N | \boldsymbol{\sigma}_N] = \frac{\exp\{-N[\frac{1}{2}t_N^2 - \Phi_N(t_N; \nu_1; \nu_2)]\}}{\sqrt{1 - \partial_{tt}^2 \Phi_N(t_N; \nu_1; \nu_2)}} [1 + O(N^{-1})] \tag{10}$$

as $N \to \infty$, where $t_N \equiv t_N(\nu_1; \nu_2)$ is the solution of

$$\partial_t[\tfrac{1}{2}t^2 - \Phi_N(t; \nu_1; \nu_2)] = 0 \tag{11}$$

3. DISTRIBUTION OF THE FLUCTUATIONS OF ν_1 AND ν_2

As we have seen in the previous section, the probability to obtain the configuration $\{s_j\}_{j=1}^N$ at $t=1$ given the configuration $\{\sigma_j\}_{j=1}^N$ at $t=0$ depends only on two macroscopic functions $\nu_1 = N_1/N$ and $\nu_2 = N_2/N$ of these configurations [see (7) and (8)]. Therefore the notation $p(\nu_1; \nu_2) \equiv \Pr[\mathbf{s}_N | \boldsymbol{\sigma}_N]$ makes sence. The distribution of the random variables N_1 and N_2 (for a fixed initial configuration $\boldsymbol{\sigma}_N$) is then given by

$$\Pr[N_1 = N\nu_1; N_2 = N\nu_2] = \#\{\mathbf{s}_N: N_1 = N\nu_1; N_2 = N\nu_2\}\ p(\nu_1; \nu_2)$$

For a fixed configuration $\{\sigma_j\}_{j=1}^N$ the number of configurations $\{s_j\}_{j=1}^N$ with given N_1 and N_2 is given by the product of binomial coefficients $C_{N_1}^{N(1+m_N)/2} C_{N_2}^{N(1-m_N)/2}$. Therefore

$$\Pr[N_1 = N\nu_1; N_2 = N\nu_2]$$
$$\sim \frac{1}{2\pi N} \frac{(1 - m_N^2)^{1/2} \exp[-N\Psi(\nu_1; \nu_2)]}{[\nu_1 \nu_2 (1 + m_N - 2\nu_1)(1 - m_N - 2\nu_2)(1 - \partial_{tt}^2 \Phi_N(t_N; \nu_1; \nu_2))]^{1/2}} \tag{12}$$

where

$$\Psi(\nu_1;\nu_2)=\frac{1}{2}t_N^2-\Phi_N(t_N;\nu_1;\nu_2)+r(\nu_1;\nu_2)$$
$$-\frac{1}{2}\log\frac{1-m_N^2}{4}-\frac{m_N}{2}\log\frac{1+m_N}{1-m_N} \tag{13}$$

and

$$r(\nu_1;\nu_2)=\nu_1\log\nu_1+\nu_2\log\nu_2+\left(\frac{1+m_N}{2}-\nu_1\right)\log\left(\frac{1+m_N}{2}-\nu_1\right)$$
$$+\left(\frac{1-m_N}{2}-\nu_2\right)\log\left(\frac{1-m_N}{2}-\nu_2\right)$$

As $N\to\infty$ the probability distributions of N_1/N and N_2/N concentrate at the unique minimum point (in fact, zero) of the function $\Psi(\nu_1;\nu_2)$—the solution of

$$\partial_{\nu_1}\Psi(\nu_1;\nu_2)=0, \qquad \partial_{\nu_2}\Psi(\nu_1;\nu_2)=0$$

One has

$$\partial_{\nu_1}\Psi(\nu_1;\nu_2)=\partial_t\left[\frac{1}{2}t^2-\Phi_N(t;\nu_1;\nu_2)\right]\Bigg|_{t=t_N(\nu_1;\nu_2)}\partial_{\nu_1}t_N(\nu_1;\nu_2)$$
$$-\partial_{\nu_1}\Phi_N(t,\nu_1,\nu_2)|_{t=t_N(\nu_1;\nu_2)}-\log\left(\frac{1+m_N}{2\nu_1}-1\right)$$

The first term in the rhs of the last expression equals zero since t_N is an extremum of $\frac{1}{2}t^2-\Phi_N(t;\nu_1;\nu_2)$. Therefore the equation $\partial_{\nu_1}\Psi(\nu_1;\nu_2)=0$ is equivalent to

$$\nu_1=\frac{1+m_N}{2}I_1\Bigg|_{t=t_N(\nu_1,\nu_2)} \tag{14}$$

Analogously, the equation $\partial_{\nu_2}\Psi(\nu_1;\nu_2)=0$ yields

$$\nu_2=\frac{1-m_N}{2}I_2\Bigg|_{t=t_N(\nu_1,\nu_2)} \tag{15}$$

Denoting by $\nu_{1,N}$ and $\nu_{2,N}$ the solutions of the system (14), (15) and substituting Eqs. (14) and (15) in Eq. (11), we obtain $t_N(\nu_{1,N};\nu_{2,N})=0$.

Using this fact and passing to the limit $N\to\infty$ in Eqs. (14) and (15), we arrive at

$$\nu_{1,2}^{*}=\frac{1\pm m_{*}}{2}\int_{0}^{\infty}\frac{d\mu}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(\mu-J_{0}m_{*})^{2}\right] \tag{16}$$

where $m_{*}=\lim_{N\to\infty}m_{N}$ and $\nu_{j}^{*}=\lim_{N\to\infty}\nu_{j,N}$, $j=1,2$. Since, as $N\to\infty$, the probabilities of $N_1=\nu N$, where $\nu\neq\nu_1^*$, are exponentially small, the limiting value of N_1/N is "nonrandom" and is equal to ν_1^*. Analogously, $\lim_{N\to\infty}N_2/N=\nu_2^*$ (in probability).

Note that *assuming* the validity of the law of large numbers for the sequence of arithmetic means

$$\frac{N_1}{N}=\frac{1}{4N}\sum_{j=1}^{N}(s_j(1)+1)(s_j(0)+1)$$

we could have obtained the typical value ν_1^* of N_1/N (and similarly ν_2^*) at once. Indeed if one can apply the law of large numbers, then $\nu_1^*=\Pr[s_k(1)=1](1+m_*)/2$. On the other hand, as $N\to\infty$, $s_k(1)=\operatorname{sgn}[\mathcal{N}(0,1)+J_0m_*]$ and therefore

$$\Pr[s_k(1)=1]=\int_{-J_0m_*}^{\infty}\frac{d\mu}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\mu^2\right)$$

which implies Eq. (16).

Equation (12) is the local limit theorem for the random vector $(N_1;N_2)$. The corresponding integral limit theorem is stated as follows. Consider two sequences of rectangles

$$\mathscr{A}_N^{(0)}=[\sqrt{N}\,a_1,\sqrt{N}\,b_1]\times[\sqrt{N}\,a_2,\sqrt{N}\,b_2]\subset\mathbf{R}^2$$

and

$$\mathscr{A}_N=\mathscr{A}_N^{(0)}+(N\nu_1^*;N\nu_2^*)\subset\mathbf{R}^2$$

Then

$$\Pr[(N_1;N_2)\in\mathscr{A}_N]\to\frac{1}{\pi c}\int_{a_1}^{b_1}\int_{a_2}^{b_2}dx_1\,dx_2\exp\left\{-\frac{Q(x_1,x_2)}{I(1-I)}\right\} \tag{17}$$

as $N \to \infty$, where

$$Q(x_1, x_2) = \frac{x_1^2}{1+m_*} + \frac{x_2^2}{1-m_*} - \frac{(x_1 - x_2)^2}{2[1 + 2\pi I(1-I)\exp(J_0^2 m_*^2)]}$$

$$c^2 = (1 - m_*^2)\, I(1-I) \left[I(1-I) + \frac{\exp(-J_0^2 m_*^2)}{2\pi} \right]$$

and

$$I \equiv \lim_{N\to\infty} I_1|_{t=0} = \lim_{N\to\infty} I_2|_{t=0}$$

The validity of the law of large numbers for the sequence q_N [see (3)] follows from the above results for N_1 and N_2. Indeed

$$Nq_N = \sum_{k=1}^{N} s_k(1)\, s_k(0) = 2(N_1 - N_2) - Nm_*$$

Therefore Eq. (16) yields

$$\lim_{N\to\infty} q_N = m_* \int_{-J_0 m_*}^{J_0 m_*} \frac{d\mu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\mu^2\right) = m_* m(1) \tag{18}$$

where $m(1) = \operatorname{erf}[J_0 m_*/\sqrt{2}]$. The limiting distribution density $\phi(\mu)$ of the random variables

$$N^{-1/2} \sum_{k=1}^{N} [s_k(1) - m(1)]\, s_k(0) = 2\left[\frac{N_1 - N v_1^*}{\sqrt{N}} - \frac{N_2 - N v_2^*}{\sqrt{N}}\right]$$

—fluctuations of q_N—can be easily found from Eq. (17):

$$\phi(\mu) = \frac{1}{2[2\pi I(1-I) + \exp(-J_0^2 m_*^2)]^{1/2}} \times \exp\left[-\frac{\mu^2}{4(2I(1-I) + \exp(-J_0^2 m_*^2)/\pi)}\right] \tag{19}$$

Note that *assuming* that the limiting distribution of the normalized sums

$$s_{1,0} \equiv N^{-1/2} \sum_{l=1}^{N} [s_l(1) - m(1)]\, s_l(0)$$

is Gaussian, one can obtain Eq. (19) easily, since the variance of these sums can be calculated as follows. One has

$$\mathbf{E}s_{1,0}^2 = 1 - m^2(1) + \frac{1}{N}\sum_{l \neq j} s_l(0)\, s_j(0)\, \mathbf{E}[s_l(1)\, s_j(1) - m^2(1)]$$

Keeping only the relevant corrections to the limiting distributions, one can represent $s_j(1)$ and $s_l(1)$ [see Eq. (2)] as follows:

$$s_j(1) = \operatorname{sgn}[\mathcal{N}_j(0, 1) + J_0 m_* + N^{-1/2} J_{jl} s_l(0)]$$
$$s_l(1) = \operatorname{sgn}[\mathcal{N}_l(0, 1) + J_0 m_* + N^{-1/2} J_{lj} s_j(0)]$$

where

$$\mathcal{N}_j(0, 1) \stackrel{d}{=} \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{k(\neq j, l) = 1}^{N} J_{j,k} s_k(0)$$

and

$$\mathcal{N}_l(0, 1) \stackrel{d}{=} \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{k(\neq j, l) = 1}^{N} J_{l,k} s_k(0)$$

are independent random variables with the standard normal distribution independent of $J_{j,l}$. Next,

$$\mathbf{E}[s_j(1)\, s_l(1)] = \mathbf{E}[\mathbf{E}[s_j(1)\, s_l(1) \mid J_{j,l}]] = \mathbf{E}[\mathbf{E}[s_j(1) \mid J_{j,l}]\, \mathbf{E}[s_l(1) \mid J_{j,l}]]$$

and

$$\begin{aligned}\mathbf{E}[s_j(1) \mid J_{j,l}] &= \operatorname{erf}\left[\frac{J_0 m_*}{\sqrt{2}} + \frac{J_{j,l} s_l(0)}{\sqrt{2N}}\right] \\ &= m(1) + J_{j,l} s_l(0) \sqrt{\frac{2}{N\pi}} \exp\left(-\frac{J_0^2 m_*^2}{2}\right) + O(N^{-1})\end{aligned}$$

where $\mathbf{E}[\cdot \mid J_{j,l}]$ is the conditional expectation given $J_{j,l}$, yield

$$\mathbf{E}[s_j(1)\, s_l(1)] = m^2(1) + \frac{2 s_j(0)\, s_l(0)}{N\pi} \exp(-J_0^2 m_*^2) + O(N^{-3/2})$$

Hence,

$$\mathbf{E}s_{1,0}^2 = 1 - m^2(1) + 2\exp(-J_0^2 m_*^2)/\pi$$

which coincides with the variance of the distribution with the density (19).

Although the "short" derivation of the distribution (19) is somewhat casual, it is a good supplement to the accurate derivation since it indicates clearly the reason why the variance of the distribution (19) is different from what one would obtain in the case of sums of independent random variables.

4. THE AVERAGE NUMBER OF FIXED POINTS

In the particular case $\boldsymbol{\sigma}_N = \mathbf{s}_N$ Eq. (10) simplifies to

$$p(\boldsymbol{\sigma}_N) \equiv \Pr[\boldsymbol{\sigma}_N \mid \boldsymbol{\sigma}_N] = \frac{\exp\{-N[\frac{1}{2}t_N^2 - \phi_N(t_N; m_N)]\}}{\sqrt{1 - \partial_{tt}^2 \phi_N(t_N; m_N)}} [1 + O(N^{-1})]$$

as $N \to \infty$, where

$$\phi_N(t_N; m_N) = \frac{1 + m_N}{2} \log I_1 + \frac{1 - m_N}{2} \log(1 - I_2)$$

and $t_N \equiv t_N(m_N)$ minimizes $\frac{1}{2}t^2 - \phi_N(t; m_N)$. Therefore the main asymptotics of the average number of the fixed points with magnetization m_N is given by

$$\begin{aligned} F_N(m_N) &= \# \left\{ \boldsymbol{\sigma}_N : \frac{1}{N} \sum_{j=1}^{N} \sigma_j = m_N \right\} p(\boldsymbol{\sigma}_N) \\ &\sim 2^N \exp\left\{ -\frac{1}{2} N \left[t_N^2 - 2\phi_N(t_N; m_N) \right.\right. \\ &\qquad \left.\left. + m_N \log\left(\frac{1 + m_N}{1 - m_N}\right) + \log(1 - m_N^2) \right] \right\} \\ &\qquad \times \left[\frac{1}{2} \pi N (1 - m_*^2)(1 - \partial_{tt}^2 \phi_N(t_N; m_N)) \right]^{-1/2} \end{aligned} \tag{20}$$

The typical shapes of the rate function

$$L(m) \equiv \lim_{N \to \infty} \frac{1}{N} \log F_N(m)$$

are plotted on Figs. 1–3. Qualitatively the evolution of $L(m)$ with J_0 is the same as the evolution of the corresponding rate functions with α (the ratio of the number of stored patterns to the total number of spins) in the Hopfield[3] and diluted Hopfield models.[8] It is believed that the SK model is equivalent to the symmetric extremely diluted Hopfield model. Therefore, apparently, the rate function $L(m)$ coincides with the corresponding rate function in the extremely diluted Hopfield model.

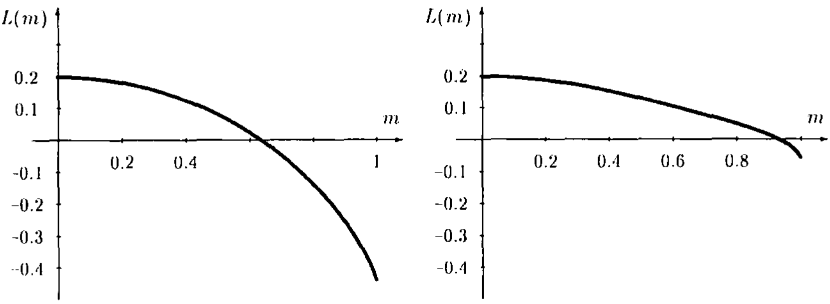

Fig. 1. The rate function $L(m)$ for the average number of the fixed points with magnetization m for $J_0=0.1$ (left) and $J_0=1.5$ (right).

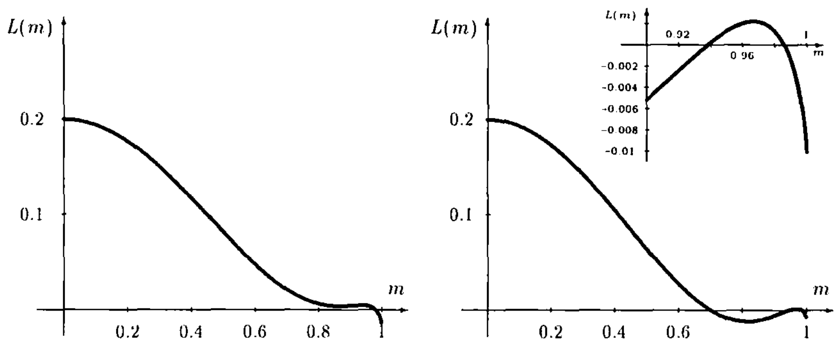

Fig. 2. The rate function $L(m)$ for the average number of the fixed points with magnetization m for $J_0=2.15$ (left) and $J_0=2.3$ (right). The inset gives the magnification of the function $L(m)$ for $J_0=2.3$ on the interval [0.9; 1.0].

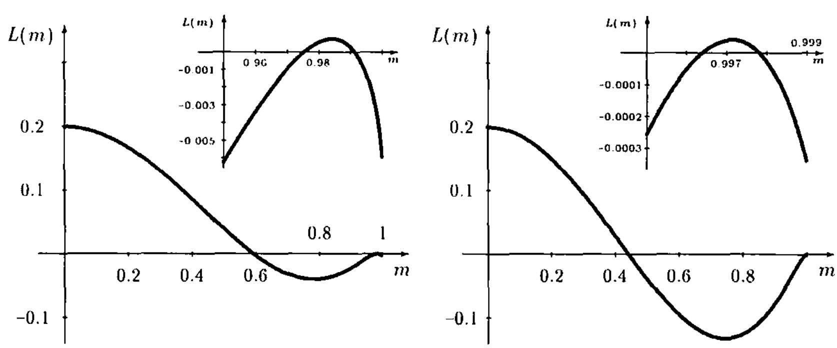

Fig. 3. The rate function $L(m)$ for the average number of the fixed points with magnetization m for $J_0=2.5$ (left) and $J_0=3.0$ (right). The insets give the magnification of the bands near $m=1$ where the rate functions are positive.

Consider the indicator functions $\chi(\sigma_N)$ given by

$$\chi(\sigma_N) = \begin{cases} 1 & \text{if } \sigma_N \text{ is a fixed point} \\ 0 & \text{otherwise} \end{cases}$$

Then the number of the fixed points of the dynamics (1) with the magnetization m_N is given by

$$\mathcal{N}(m_N) = \sum_{\sigma_N : m_N} \chi(\sigma_N)$$

where the summation runs over all configurations σ_N such that $\sum_{j=1}^{N} \sigma_N = Nm_N$. The standard application of the Chebyshev inequality

$$\Pr[\mathcal{N}(m_N) > 1/2] \leqslant 2\mathbf{E}\mathcal{N}(m_N) = 2F_N(m_N)$$

and the Borel–Cantelli lemma allows one to conclude that with probability one $\mathcal{N}(m_N) = 0$, for N large enough, whenever $m_N \to m$ and $L(m) < 0$.

Note that the last result is supplementary to (an analog of) Newman's theorem,[6] which certainly can be proven for the SK model with the ferromagnetic component and sequential version of dynamics (1). The fixed points of the sequential and the parallel dynamics coincide, so all the above results concerning the fixed points apply for the sequential dynamics as well. Loosely, (the analog of) Newman's theorem says that if J_0 is large enough, then the configuration $\mathbf{u}_N \equiv \{s_j = 1\}_{j=1}^{N}$ is in the basin of attraction of a fixed point $\mathbf{s}_N^*$ such that $N^{-1} \sum_{j=1}^{N} s_j^*$ is close to 1, or, in other words, the Hamming distance $d_H(\mathbf{u}_N, \mathbf{s}_N^*) = N^{-1} \sum_{j=1}^{N} |s_j^* - 1|$ is close to zero. The rate function $L(m)$ is always (that is, for any $J_0 \geqslant 0$) negative in a sufficiently small vicinity of $m = 1$; therefore the maximal solution of $L(m) = 0$ on the interval (0, 1) (see Fig. 3) provides an upper bound for $d_H(\mathbf{u}_N, \mathbf{s}_N^*)$. When J_0 is small the rate function $L(m)$ is negative in a rather large interval containing $m = 1$ (see Fig. 1). Therefore, one can conclude that all fixed points of the dynamics (1) are sufficiently far from the configuration $\mathbf{u}_N$ if J_0 is small.

Note that various arguments (the most reliable of which are those based on computer simulations; see, e.g., ref. 4 and references therein) predict the existence of a fixed point $\mathbf{s}_N^*$ with a basin of attraction containing $\mathbf{u}_N$ such that $m^* \equiv \lim_{N \to \infty} N^{-1} \sum_{j=1}^{N} s_j^* > 0$ (m^* is usually called the remnant magnetization). Therefore the lower bound for the Hamming distance $d_H(\mathbf{u}_N, \mathbf{s}_N^*)$ (always less than 1/2) provided by solution of $L(m) = 0$ for small J_0 is qualitatively correct. That is, the qualitatively stronger statement $d_H(\mathbf{u}_N, \mathbf{s}_N^*) \to 1/2$ as $N \to \infty$ is apparently incorrect for any $J_0 \geqslant 0$.

The average number of all fixed points is, of course, $\sum_{m_N} F(m_N)$, which for large N is well approximated by $\max_{m_N} F_N(m_N) = F_N(0)$. Therefore the asymptotics of the average number of all fixed points is the same as that in the SK model without the ferromagnetic component ($J_0 = 0$) $\lim_{N\to\infty} \log F_N/N = 1.992$.[7] Note that contrary to the (incorrect) conclusion of ref. 2, the $L(0)$ does not depend on J_0. That is, the average number of (all) the fixed points is virtually independent of the magnitude of the ferromagnetic interaction J_0.

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REFERENCES

1. J. M. G. Amaro de Matos, A. E. Patrick, and V. A. Zagrebnov, Random infinite-volume Gibbs states for the Curie–Weiss random field Ising model, *J. Stat. Phys.* **66**:139–164 (1992).
2. D. C. Dean, On the metastable states of the zero-temperature SK model, *J. Phys. A: Math. Gen.* **27**:L899–L905 (1994).
3. E. Gardner, Structure of metastable states in the Hopfield model, *J. Phys. A: Math. Gen.* **19**:L1047–L1052 (1986).
4. E. Gardner, B. Derrida, and P. Mottishaw, Zero temperature parallel dynamics for infinite range spin glasses and neural networks, *J. Phys.* (Paris) **48**:741–755 (1987).
5. S. Kirkpatrick and D. Sherrington, Infinite-ranged models of spin glasses, *Phys. Rev. B* **17**:4384–4403 (1978).
6. C. M. Newman, Memory capacity in neural networks model: Rigorous lower bounds, *Neural Networks* **1**:223–238 (1988).
7. F. Tanaka and S. F. Edwards, *J. Phys. F* **10**:2471 (1980).
8. A. Treves and D. J. Amit, Metastable states in asymmetrically diluted Hopfield networks, *J. Phys. A: Math. Gen.* **21**:3155–3169 (1988).